JOURNAL OF APPROXIMATION THEORY 19, 332–360 (1977)

LM-g Splines*

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Communicated by I. J. Schoenberg

Received November 6, 1975

As an extension of the notion of an L-g spline, three mathematical structures called LM-g splines of types I, II, and III are introduced. Each is defined in terms of two differential operators

$$L = D^n + \sum_{j=0}^{n-1} a_j(t)D^j \quad \text{and} \quad M = \sum_{i=0}^{m} b_i(t)D^i, \text{where } n \geqslant m \geqslant 0,\ n > 0,\ D = d/dt,$$

the coefficients a_j, $j = 0,..., n-1$, and b_i, $i = 0,..., m$, are sufficiently smooth; and b_m is bounded away from zero on $[0, T]$. Each of the above types of splines is the solution of an optimization problem more general than the one used in the definition of the L-g spline and hence it is recognized as an entity which is distinct from and more general mathematically than the L-g spline. The LM-g splines introduced here reduce to an L-g spline in the special case in which $m = 0$ and $b_0 = \text{constant} \neq 0$. After the existence and uniqueness conditions, characterization, and best approximation properties for the proposed splines are obtained in an appropriate reproducing kernel Hilbert space framework, their usefulness in extending the range of applicability of spline theory to problems in estimation, optimal control, and digital signal processing are indicated. Also, as an extension of recent results in the generalized spline literature, state variable models for the LM-g splines introduced here are exhibited, based on which existing least squares algorithms can be used for the recursive calculation of these splines from the data.

1. INTRODUCTION

As a generalization of the L-g spline function [1, 2], we introduce in the present paper three types of generalized splines, which we call *LM-g splines* because they are defined in terms of two operators L and M. An LM-g spline reduces to an L-g spline if M is the identity operator.

Our basic motivation for introducing LM-g splines into spline theory is that they permit a wider application of this theory to problems in estimation

* Supported by NSF Grant ENG 74-17955.

ISSN 0021-9045

of random processes, minimum-energy controls, digital signal processing, and systems modeling.

In the remaining part of this section, we present a brief introduction to the results in this paper.

In an *LM-g* spline, L and M are linear differential operators, of degrees n and $m \leqslant n$ respectively, with coefficients not necessarily constant, the domains of L and M being appropriate Sobolev spaces of real-valued functions on an interval $[0, T]$ of the real line.

In what follows, G_M will denote a suitably defined Green's function pertaining to M. For each of the three types of *LM-g* splines to be defined, let Y denote the space over a subset of which the minimization defining the spline is to be carried out. Specifically, Y will be the domain of G_M, L, and M, respectively, for *LM-g* splines of types I, II, and III. Also, in each case, let Φ denote a set of k continuous linear functionals on Y, r a k-tuple of real numbers, and $U(r)$ the set consisting of the elements of Y which interpolate or smooth r with respect to Φ. Then we say that $f^* \in Y$ is an *LM-g spline of type I* if it minimizes $\| L\, G_M f \|$, an *LM-g spline of type II* if it minimizes $\| G_M\, L f \|$, and an *LM-g spline of type III* if it minimizes $\| L\, M f \|$, in each case over all f belonging to an appropriate set $U(r)$ defined as above. Also, in each case $\| \cdot \|$ denotes the norm in the space to which the range of the composition of the two operators acting on f belongs.

We will see that an *LM-g* spline of type I is simply the image of an *L-g* spline under the differential operator M; that an *LM-g* spline of type III is an image of an *L-g* spline under G_M; and that if L and M are constant coefficient operators and under appropriate boundary conditions in the definition of G_M, the *LM-g* splines of types I and II are one and the same function.

It may be remarked at this point that the three types of *LM-g* splines, which we are introducing here, have, in their respective reproducing kernel Hilbert spaces, optimal properties of "conventional splines." For this reason, it pays to think of them as "functions" rather than linear functionals indexed on t.

Of the previous work in this area, some of the contributions of Wahba and associates [3–7] are most relevant to the material presented in this paper. In particular, the generalized splines associated with spaces of functions with rapidly decreasing Fourier transforms, referred to by Wahba [6], constitute specific instances of the *LM-g* splines defined by us. Also, very relevant to the present work are the results of Weinert and associates [8–11] and of the author and Caprihan [17].

In what follows, we present, in Sections 2 through 4, a detailed formulation of the *LM-g* interpolating and smoothing splines of type I, in an appropriate reproducing kernel Hilbert space framework. In Section 5, as an extension of existing results [3, 10] for the *L*-spline, we construct a state variable stochastic model such that the LM-g interpolating and smoothing splines of type I are least squares estimates of the model ouput given the data; and, on this basis, we indicate how least-squares algorithms may be used for nonrecursive and recursive calculation of these splines.

In sections 6 and 7, we summarize for the LM-g splines of types II and III results similar to the ones obtained for the LM-g spline of type I. A detailed treatment is omitted since it would follow along parallel lines.

In Section 8, we consider the special case in which the differential operators L and M have constant coefficients.

Finally, in Section 9, we discuss applications of LM-g splines to the problems in signal and system theory mentioned above.

It goes without saying that the results presented in this paper extend trivially to the case in which L and M are abstract operators in a Banach space rather than differential operators as assumed by us. Since such a generalization is unnecessary for the types of applications envisaged here, it is not discussed in the present paper.

2. The LM-g Interpolating Spline of Type I

For n a positive integer and t a variable belonging to an interval $[0, T]$ of the real line, let

$$L = D^n + \sum_{j=0}^{n-1} a_j(t)\, D^j, \tag{1}$$

and

$$M = \sum_{i=0}^{m} b_i(t)\, D^i, \tag{2}$$

where $D = d/dt$, $0 \leqslant m \leqslant n$, $n > 0$, $a_j \in C^j\ [0, T]$, $j = 0,\ldots, n-1$; $b_i \in C^{i+n-m}\ [0, T]$, $i = 0,\ldots, m$, and b_m is bounded away from zero on $[0, T]$.

If P is an operator from a space X to a space Y, we will denote its null space by $N(P)$ and its range by PX.

Let H^j, with j a nonnegative integer, denote the linear space of real-valued functions f on $[0, T]$ such that $f^{(j-1)}(\equiv D^{j-1}f)$ is absolutely continuous and $f^{(j)} \in L^2(0, T)$ (where $L^2(0, T) \equiv H^0$ denotes the linear space of square integrable real functions on $(0, T)$). H^j is a Banach space under any of the equivalent Sobolev norms of the form $\| g \| = \{\sum_{i=1}^{j} (F_i g)^2 + \int_0^T (Pg(t))^2\, dt)^{1/2}$, where $g \in H^j$, P is a linear differential operator of order j from H^j to H^0 (P being in the form expressed by the right side of (1) with n replaced by j), and F_i, $i = 1,\ldots,$ are linear functionals on H^j which are linearly independent on $N(P)$. Let $\mathscr{F}^j$ denote the set of linear functionals F on H^j of the form $Fg = \sum_{i=0}^{j} \int_0^T D^i g(t)\, d\mu_i(t)$, where μ_i are functions of bounded variation.

Let us introduce the subspace

$$N_2 = N(L) \cap N(M) \tag{3}$$

and denote by n_2 its dimension.

Also, let $\psi = \{\psi_1, ..., \psi_m\}$ be a set of linear functionals belonging to $\mathscr{F}^n$ which are linearly independent on $N(M)$, and, for convenience, assume that these functionals are labeled so that the first n_2 of them are linearly independent on N_2. It is then possible to write $N(L)$ and $N(M)$ in the form

$$N(L) = N_1 \oplus N_2, \tag{4}$$

$$N(M) = N_2 \oplus N_3, \tag{5}$$

where $\oplus$ denotes direct sum, and

$$N_1 = \{g \in N(L) \colon \psi_i g = 0,\ i = 1, ..., n_2\}, \tag{6a}$$

$$N_3 = \{g \in N(M) \colon \psi_i g = 0,\ i = 1, ..., n_2\}, \tag{6b}$$

and the dimensions n_1 and n_3 of N_1 and N_3 clearly satisfy $n_1 = n - n_2$ and $n_3 = m - n_2$.

In addition, by means of ψ it is possible to define an inverse G_M of the restriction of M to the subspace

$$H_\psi^n = \{g \in H^n \colon \psi_i g = 0,\ i = 1, ..., m\}. \tag{7}$$

Specifically, this inverse is the Green's function $G_M(t, u)$ of the problem

$$Mg = f, \qquad \psi_i g = 0, \qquad i = 1, ..., m. \tag{8}$$

G_M may be constructed by the procedure in [1, pp. 959–960].

For simplicity, we will abbreviate integral operator actions as follows:

$$\int_0^T G_M(t, u) f(u)\, du \equiv G_M(t, \cdot) \circ f(\cdot) \equiv G_M(t, \cdot) \circ f, \tag{9}$$

and we will further abbreviate the function $G_M(\cdot, \cdot) \circ f(\cdot)$ (i.e., the set of all t-evaluations of (9)) by $G_M f$.

Assume finally that we are given a set $\Phi = \{\phi_1, ..., \phi_k\}$, where $k \geqslant n_1$, of linear functionals in $\mathscr{F}^{n-m}$, which are linearly independent on MH_ψ^n.

We have:

DEFINITION 1. Let L, M, Ψ, G_M, and Φ be as above. Given a k-tuple $r = (r_1, ..., r_k)$ of real numbers, an *LM-g spline* $S_I(L, M, \Psi, \Phi, r; \cdot)$ *of type I interpolating r with respect to* Φ, is defined by the minimization[1]

$$\min_{f \in U_1(r)} \int_0^T (L_{(t)} G_M(t, \cdot) \circ f)^2 \, dt$$

$$= \int_0^T (L_{(t)} G_M(t, \cdot) \circ S_I(L, M, \Psi, \Phi, r; \cdot))^2 \, dt, \tag{10}$$

where

$$U_1(r) = \{f \in MH_\psi{}^n: \phi_i f = r_i, i = 1, ..., k\}.^2 \quad \| \tag{11}$$

Remark 1. If the functionals ϕ_i, $i = 1, ..., k$, are evaluation functionals, that is $\phi_i f = f(t_i)$, $t_i \in [0, T]$, $i = \ ,..., k$, we call the corresponding spline simply an *LM spline of type I.* ||

Remark 2. If the operator M is of zero order, then the corresponding *LM-g* spline of type I reduces to the *L-g* spline of Jerome and Schumaker [2]. ||

3. REPRODUCING KERNEL HILBERT SPACE RESULTS FOR THE TYPE I *LM-g* INTERPOLATING SPLINE

3.1. *Brief Review of Pertinent L-g Spline Results*

It is necessary at this point to recall briefly the definition of the *L-g* interpolating spline and the structure of the reproducing kernel Hilbert space in which it appears as the solution of a minimum norm problem. Let L be the differential operator previously defined and suppose $\Gamma = \{\gamma_1, ..., \gamma_l\}$, $l \geqslant n$, is a set of linearly independent functionals in $\mathscr{F}^n$.

DEFINITION 2. Given a real l-vector $q = \text{col}(q_1, ..., q_l)$, the *L-g spline* $\tilde{S}(L, \Gamma, q; \cdot)$ interpolating q with respect to Γ is defined by the minimization

$$\min_{\tilde{f} \in V(q)} \int_0^T (L_{(t)} \tilde{f}(t))^2 \, dt = \int_0^T (L_{(t)} \tilde{S}(L, \Gamma, q; t))^2 \, dt, \tag{12}$$

[1] A subscript in parentheses on a symbol representing an operator or functional indicates the variable with respect to which the operation is performed.

[2] The end of a formal statement such as a definition, a theorem, or a proof, will be signaled by the symbol ||.

where

$$V(q) = \{\tilde{f} \in H^n: \gamma_i \tilde{f} = q_i,\ i = 1,..., l\}. \quad \|$$ (13)

It has been shown by de Boor and Lynch [1] and others [2, 8, 12, 13], that the solution $\tilde{S}(L, \Gamma, q; \cdot)$ always exists; and it is unique if and only if

$$N(L) \cap V(O) = \Theta, \tag{14}$$

where Θ denotes the null subspace. By a well-known argument [2, 8], condition (14) may be shown to be equivalent to the requirement that n of the elements of Γ, say $\gamma_1,..., \gamma_n$, be linearly independent on $N(L)$. de Boor and Lynch [1] showed further that if $\{\xi_i : i = 1,..., n\}$ constitute the basis for $N(L)$ dual to $\gamma_1,..., \gamma_n$, that is, if ξ_i, $i = 1,..., n$, are solutions of

$$L\xi_i = 0, \qquad \gamma_i \xi_j = \delta_{ij}, \qquad i, j = 1,..., n, \tag{15}$$

then

$$\langle\langle \tilde{f}, \tilde{g} \rangle\rangle \equiv \sum_{i=1}^{n} (\gamma_i \tilde{f})(\gamma_i \tilde{g}) + \int_0^T (L\tilde{f}(t))(L\tilde{g}(t))\, dt \qquad \forall \tilde{f}, \tilde{g} \in H^n, \tag{16}$$

is an inner product in H^n, which makes H^n a reproducing kernel Hilbert space $\tilde{H}^n$ with the reproducing kernel

$$\tilde{K}(t, u) = \sum_{i=1}^{n} \xi_i(t)\, \xi_i(u) + \int_0^T G_L(t, v)\, G_L(u, v)\, dv, \tag{17}$$

where G_L is the Green's function of the problem

$$Lf = W, \qquad \gamma_i f = 0, \qquad i = 1,..., n. \tag{18}$$

As emphasized by Weinert [8], the *L-g* spline of Definition 2 is the solution of the minimum norm problem in $\tilde{H}^n$

$$\min_{\tilde{f} \in V(q)} |||\tilde{f}|||^2 = |||\tilde{S}(L, \Gamma, q; \cdot)|||^2, \tag{19}$$

where $||| \cdots |||$ denotes the norm in $\tilde{H}^n$ induced by the inner product (16).

3.2. *On the Type I LM-g Interpolating Spline*

Returning to our original problem, we are now able to formulate the following.

THEOREM 1. *Suppose L, M, Ψ, Φ, and r are as in Definition* 1. *Then*

$$S_I(L, M, \Psi, \Phi, r; t) = M_{(t)} \tilde{S}(L, \Gamma, q; t), \tag{20}$$

provided we pick l, $\Gamma = \{\gamma_1, \ldots, \gamma_l\}$, *and* $q = (q_1, \ldots, q_l)$ *as follows*:

$$l = m + k; \tag{21}$$

$$\gamma_i = \phi_i M, \qquad i = 1, \ldots, n_1, \tag{22a}$$

$$= \psi_{i-n_1}, \qquad i = n_1 + 1, \ldots, n_1 + m, \tag{22b}$$

$$= \phi_{i-m} M, \qquad i = n_1 + m + 1, \ldots, (m + k); \tag{22c}$$

$$q_i = r_i, \qquad i = 1, \ldots, n_1, \tag{23a}$$

$$= 0, \qquad i = n_1 + 1, \ldots, n_1 + m, \tag{23b}$$

$$= r_{i-m}, \qquad i = n_1 + m + 1, \ldots, (m + k). \quad \| \tag{23c}$$

Proof. Since $U_1(r)$ is in the range of M, and M is one-to-one and onto from $H_\psi{}^n$ to $HM_\psi{}^n$, $U_1(r)$ is isomorphic under M to the set

$$\{\tilde{f} \in MH_\psi{}^n \colon M\tilde{f} \in U_1(r)\} = \{\tilde{f} \in H^n \colon \psi_i \tilde{f} = 0, i = 1, \ldots, m, \tag{24a}$$

$$\phi_j M\tilde{f} = r_j, j = 1, \ldots, k\} \tag{24b}$$

$$= \{\tilde{f} \in H^n \colon \gamma_i \tilde{f} = q_i, i = 1, \ldots, k + m\} \tag{24c}$$

$$= V(q), \tag{24d}$$

(24c) and (24d) following from (22a), (22b), (22c) and (23a) (23b), and (23c).

Let f and $\tilde{f}$ denote the corresponding elements of $U_1(r)$ and $\hat{V}(q)$, i.e.,

$$f = M\tilde{f}, \qquad f \in U_1(r), \tilde{f} \in \hat{V}(q), \tag{25a}$$

or equivalently,

$$\tilde{f} = G_M f, \qquad f \in U_1(r), \tilde{f} \in \hat{V}(q). \tag{25b}$$

According to (25b),

$$L_{(t)} G_M(t, \cdot) \circ f = L_{(t)} \tilde{f}(t). \tag{26}$$

Substitution of (26) in (10) shows that the minimization problem (10) is equivalent to the minimization problem (12), provided $V(q)$ is chosen as in (24a–d), with the associated correspondence (25a), (25b). In particular, it follows from (25a) that the minimizers of the two problems are related by (20). ||

THEOREM 2. *The LM-g spline of Definition* 1 *always exists.* ||

Proof. Clear, since the *L-g* spline $\tilde{S}(L, \Gamma, q; \cdot)$ in (20) always exists [2]. ||

THEOREM 3. *The LM-g spline of Definition* 1 *is unique if and only if*

$$W \cap U_1(O) = \Theta, \tag{27}$$

where $W \equiv MN(L) = MN_1$. ||

Proof. Since by (22b) and (23b), (27) is equivalent to (14), (27) implies uniqueness of $\tilde{S}(L, \Gamma, q; \cdot)$ in (20) and hence, of $S_I(L, M, \Psi, \Phi, r; \cdot)$. To prove the "only if" part of our assertion, suppose the solution $y^* \equiv S_I(L, M, \Psi, \Phi, r; \cdot)$ of the minimization problem (10) is unique and yet (27) is not required to hold. Then there is at least one nonzero element, say z_0, in $W \cap U_1(O)$. Substitution of $y_1^* \equiv y^* + z_0$ and $y_2^* \equiv y^* - z_0$ in (10), and (11) shows that y^*, y_1^* and y_2^* are solutions of the minimization problem (10), which contradicts our assumption that y^* is unique. ||

Henceforth, we shall assume that the conditions of Theorem 3 hold and hence that n_1 of the functionals ϕ_i, $i = 1,..., k$, specifically $\phi_1, ..., \phi_{n_1}$, are linearly independent on W.

The following gives the reproducing kernel Hilbert space structure for the spline under construction.

THEOREM 4. *Under* (27),

$$\langle f, g\rangle = \sum_{i=1}^{n_1} (\phi_i f)(\phi_i g) + \int_0^T (L_{(t)} G_M(t, \cdot) \circ f)(L_{(t)} G_M(t, \cdot) \circ g)\, dt \tag{28}$$

is an inner product for all f and $g \in MH_\psi{}^n$, *which makes* $MH_\psi{}^n$ *a reproducing kernel Hilbert space, denoted henceforth by* H_1^{n-m}, *with the reproducing kernel*

$$K(t, u) = \sum_{i=1}^{n_1} \eta_i(t)\, \eta_i(u) + \int_0^T M_{(t)} M_{(u)} G_L(t, v)\, G_L(u, v)\, dv, \tag{29}$$

where η_i, $i = 1,..., n_1$, *are the elements of the basis for W dual to* ϕ_i, $i = 1,..., n_1$, *that is, (according to* (25a))

$$\eta_i = M\xi_i, \qquad i = 1,..., n_1, \tag{30}$$

with ξ_i, $i = 1,..., n_1$, *defined by* (15), *together with* (22a), (22b). ||

Proof. Because $H_\psi{}^n$ is a closed subspace of H^n, the restrictions $\langle\langle \tilde{f}, \tilde{g}\rangle\rangle$ and $\tilde{K}(t, u)$ of (16) and (17) to f and $g \in H_\psi{}^n$ are valid inner product and reproducing kernel for $H_\psi{}^n$. Since M is continuous, one-to-one and onto, from $H_\psi{}^n$ to $MH_\psi{}^n$, (28) is obtained from (16) by setting $f = M\tilde{f}$ and $g = M\tilde{g}$ (or equivalently, $\tilde{f} = G_M f$ and $\tilde{g} = G_M g$) and using (22a), while (29) is obtained by the well-known technique of replacing the action of an operator on the elements of a reproducing kernel Hilbert space by its action on the reproducing kernel, i.e., (29) follows from[3]:

$$K(t, u) = M_{(t)} M_{(u)} \tilde{K}(t, u). \quad \| \tag{31}$$

As in the case of (19), Definition 1 may be reformulated in the reproducing kernel Hilbert space H_1^{n-m}. Thus if $\| \cdot \|$ denotes the norm in this space (induced by (28)), we have:

DEFINITION 3. If L, M, Ψ, Φ, and r are as before, $S_l(L, M, \Psi, \Phi, r; \cdot)$ is the solution of

$$\min_{f \in U_1(r)} \|f\|^2. \quad \| \tag{32}$$

Equation (32) is a conventional minimum norm problem in the Hilbert space H_1^{n-m}. The solution for such a problem is well known [15]: it is the unique element f_0 ($\equiv S_l(L, M, \Psi, \Phi; \cdot)$) of H_1^{n-m} orthogonal to $U_1(0)$, that is, lying in the span of the representers h_i, $i = 1,..., k$, of the linear functionals ϕ_i, $i = 1,..., k$, in H_1^{n-m}, satisfying the data constraints. Specifically,

$$f_0(t) = \sum_{j=1}^{k} \alpha_j h_j(t), \tag{33}$$

where, by a well-known property of representation of linear functionals in a reproducing kernel Hilbert space [14, 8],

$$h_j(u) = \phi_{j(t)} K(t, u), \qquad j = 1,..., k, \tag{34}$$

and the constants α_j, $j = 1,..., k$, are determined from the requirement that

$$\phi_i f_0 = \sum_{j=1}^{k} \langle h_i, h_j \rangle \alpha_j = r_i, \qquad i = 1,..., k. \tag{35}$$

As in the L-g spline case [8], the solution of (35) permits us to express $S_l(L, M, \Psi, \Phi, r; t)$ explicitly in terms of the data.

[3] By a straightforward but rather tedious calculation, one can show that $K(t, u)$ defined by (29) satisfies the two requirements for it to be the reproducing kernel of H_1^{n-m}, namely, that $K(\cdot, u)$ is an element of H_1^{n-m} and $\langle K(\cdot, u), f(\cdot) \rangle = f(u) \ \forall f \in H_1^{n-m}$ (for basic theory of reproducing kernel Hilbert spaces see [14]).

In fact, with the notation

$$\alpha = \text{col}(\alpha_1, \ldots, \alpha_k), \tag{36}$$

$$h = \text{col}(h_1, \ldots, h_k), \tag{37}$$

$$H = k \times k \quad \text{matrix with the } (ij)\text{th element} = \langle h_i, h_j \rangle, \tag{38}$$

$$r = \text{col}(r_1, \ldots, r_k), \tag{39}$$

Eqs. (35) take the form

$$H\alpha = r, \tag{40}$$

which, since H is invertible because h_i, $i = 1, \ldots, k$, are linearly independent, leads to

$$f_0(t) \equiv S_I(L, M, \Phi, r; t) = h^T(t)\, H^{-1} r, \tag{41}$$

where the superscript T denotes the transpose.

Before closing this section, it is worthwhile in connection with the applications of Section 9, to state the property of "best approximation of linear functionals" for the LM-g spline approximation under discussion. This property holds, of course, in the present case, on the basis of well-known arguments developed for splines in a Hilbert space by Golomb and Weinberger [16], and hence it is stated without proof.

THEOREM 5 (Best Approximation of Linear Functionals). *For ϵ a positive constant and $U_1(r)$ as in* (11) *let*

$$\hat{U}_1(r) = \{f \in U_1(r) \colon \|f\|^2 \leqslant \epsilon^2\}, \tag{42}$$

and assume that $\hat{U}_1(r)$ is nonempty. Then, given a continuous linear functional χ on H_1^{n-m}, the value $\chi(f)$ which minimizes

$$\sup_{g \in \hat{U}_1(r)} |\chi(f) - \chi(g)|^2 \tag{43}$$

over all $f \in H_1^{n-m}$ is given by $\chi(f_0)$ where $f_0 = S_I(L, M, \Psi, \Phi, r; \cdot)$. ||

4. THE LM-g SMOOTHING SPLINE OF TYPE I

In a manner analogous to its interpolating counterpart, an *LM-g smoothing spline of Type I* may be defined as a generalization of the *L-g smoothing spline*. Suppose L, M, Ψ, and Φ are as before, and let there be given the real data vector $r = \mathrm{col}(r_1, \ldots, r_k)$ and a symmetric positive definite $k \times k$ matrix Q which expresses the fidelity that the solution is required to maintain to the data.

DEFINITION 4. An *LM-g smoothing spline* $S_I(L, M, \Psi, \Phi, r; Q; \cdot)$ *of Type I* is the solution of the unconstrained minimization problem:

$$\min_{f \in MH_\psi{}^n} \left\{ \int_0^T (L_{(t)} G_M(t, \cdot) \circ f)^2 \, dt + (r - \Phi f)^T Q^{-1} (r - \Phi f) \right\}, \tag{44}$$

where

$$\Phi f = \mathrm{col}(\phi_1 f, \ldots, \phi_k f). \quad \| \tag{45}$$

The following points may be made regarding the above spline:

(i) By an argument similar to that in the preceding section, it follows that, with Γ and q as described by (22a–c) and (23a–c),

$$S_I(L, M, \Psi, \Phi, r; Q; t) = M_{(t)} \tilde{S}(L, \Gamma, q; Q; t). \tag{46}$$

Here, $\tilde{S}(L, \Gamma, q; Q; \cdot)$ is the *L-g* smoothing spline, defined as the solution of the minimization problem:

$$\min_{\substack{\tilde{f} \in H_\psi{}^n \\ \psi_i \tilde{f} = 0, i = 1, \ldots, m}} \left\{ \int_0^T (L\tilde{f}(t))^2 \, dt + (q - \Gamma \tilde{f})^T Q^{-1} (q - \Gamma \tilde{f}) \right\} \tag{47}$$

where

$$\Gamma \tilde{f} = \mathrm{col}(\gamma_1 \tilde{f}, \ldots, \gamma_k \tilde{f}). \tag{48}$$

We will assume that (27) holds. The existence and uniqueness of $S_I(L, M, \Psi, \Phi, r; Q; \cdot)$ then follows from the existence and uniqueness of $\tilde{S}(L, \Gamma, q; Q; \cdot)$.

(ii) Introduce the Hilbert space $W = L^2 \times R^k$ with the inner product in W defined by

$$(f, g)_W = (f_1, g_1)_{L^2} + f_2{}^T Q^{-1} g_2, \tag{49}$$

where $f = \mathrm{col}(f_1, f_2)$, $g = \mathrm{col}(g_1, g_2)$, with $f_1, g_1 \in H_1^{n-m}$ and $f_2, g_2 \in R^k$.

Let $h_j, j = 1, \ldots, k$, be as in (34) and define $\pi: H_1^{n-m} \to L^2$, $\Phi: H_1^{n-m} \to R^k$, $\Phi^\dagger$, $\tilde{\pi}: H_1^{n-m} \to W$, $\tilde{\pi}^\dagger$, and $p \in W$ by

$$\begin{aligned} &\pi = LG_M; \quad \Phi = \mathrm{col}(\langle h_1, \cdot \rangle, \ldots, \langle h_k, \cdot \rangle); \quad \Phi^\dagger = (h_1, \ldots, h_k); \\ &\tilde{\pi} = \mathrm{col}(P, \Phi); \quad \tilde{\pi}^\dagger = (\pi^\dagger, \Phi^\dagger); \quad \pi^\dagger = \mathrm{Adj}(\pi); \quad p = \mathrm{col}(0, r). \end{aligned} \tag{50}$$

Finally, for simplicity in notation let $f^* \equiv S_I(L, M, \Psi, \Phi, r; Q; \cdot)$.

Then, the functional to be minimized in (34) may be rewritten as

$$J(f) = (\tilde{\pi} \in -p, \tilde{\pi}f - p)_W, \tag{51}$$

and following the developments in [17] we have Theorems 6 and 7 below.

In fact, taking the Gateaux differential of (51) along $\Delta f \in H_1^{n-m}$,

$$2\langle \tilde{\pi}^\dagger \tilde{\pi} f - \tilde{\pi}^\dagger p, \Delta f \rangle, \tag{52}$$

the requirement that (52) vanish for all Δf at $f = f^*$ leads to

THEOREM 6. *f^* is the solution of*

$$\tilde{\pi}^\dagger \tilde{\pi} f = \tilde{\pi}^\dagger p. \quad \| \tag{53}$$

Let Λ denote the orthogonal complement of $N(\Phi)$ in H_1^{n-m}; that is, Λ is the span of $\{h_1, ..., h_k\}$.

THEOREM 7. $f^* \in \Lambda$.

Proof. Let μ denote the orthogonal projection operator from H_1^{n-m} to Λ. Then

$$\begin{aligned} J(f) &= (\pi[\mu f + (f - \mu f)], \pi[\mu f + (f - \mu f)])_{L^2} + (\Phi\mu f - p)^T Q^{-1}(\Phi\mu f - p) \\ &= (\pi\mu f, \pi\mu f)_{L^2} + (\pi[f - \mu f], \pi[f - \mu f])_{L^2} + (\Phi\mu f - p)^T Q^{-1}(\Phi\mu f - p) \\ &= J(\mu f) + \|f - \mu f\|^2_{H_1^{n-m}} \\ &\geqslant J(\mu f) \end{aligned}$$

with equality if and only if $f = \mu f$. $\|$

(iii) As done with the derivation of (41) in the interpolating case, it is possible, on the basis of the preceding, to derive a nonrecursive algorithm for obtaining f^*. Thus by Theorem 7, we may write

$$f^*(t) = \sum_{j=1}^{k} \alpha_j h_j(t) \equiv h^T(t)\,\alpha, \tag{54}$$

which when substituted in (53), use being made of (50), leads to

$$\sum_{j=1}^{k} \alpha_j [\pi^\dagger \pi + \Phi^\dagger \Phi]\, h_j = \Phi^\dagger r. \tag{55}$$

Obtaining the inner product in H_1^{n-m} of both sides of (55) by h_i, $i = 1, ..., k$, and expressing the resulting k equations in matrix form, we get

$$(B + H^T Q^{-1} H)\,\alpha = H^T Q^{-1} r, \tag{56a}$$

where H is defined by (38) and B is a matrix with elements B_{ij}, $i, j, \ldots, k$, given by

$$B_{ij} = (\pi h_i, \pi h_j)_{L^2} = \Phi_{i(s)} \Phi_{j(t)} K_1(s, t), \tag{56b}$$

where K_1 is defined by (59b).

Equations (56a) and (54) then give the desired nonrecursive expression for the smoothing LM-g spline of type I:

$$f^*(t) = h^T(t)(B + H^T Q^{-1} H)^{-1} B^T Q^{-1} r. \tag{57}$$

5. LM-g Splines of Type I in the Context of Estimation of Stochastic Processes

5.1. *Introduction*[4]

In the correspondence between Bayesian estimation and spline interpolation established by Kimeldorf and Wahba [3–5] and Weinert and Kailath [8, 9], the data are modeled as arising from nonnoisy discrete linear measurements made on a realization (sample function) $y = \{y(t): 0 \leqslant t \leqslant T\}$ of a second-order zero-mean real-valued stochastic process $Y = \{Y(t): 0 \leqslant t \leqslant T\}$, and the least-squares (minimum variance) estimate $\hat{y}(t)$ of $y(t)$, given the measurements, is the value at t of an L- or L-g spline *interpolating* the data. In such a correspondence, the reproducing kernel of the Hilbert space on which the spline is defined is equated to the covariance of the process Y, which permits the determination of the differential operator L, associated with the spline, from the covariance; and the functionals constraining the spline are the same as the ones expressing the measurements on the realization $\{y(t): 0 \leqslant t \leqslant T\}$.

If the measurements are contaminated by noise, independent of Y, then the least-squares estimate $\hat{y}(t)$ is the value at t of an L- or L-g *smoothing* spline, with the operator L and the constraining linear functionals same as in the interpolating case, and the fidelity matrix Q equated to the covariance of the discrete measurement noise.

Weinert and Sidhu [10] have further explored the above correspondence by deriving a state-variable model for the process Y, for the case of interpolation by L splines, and using this model to apply existing least-squares recursive smoothing techniques to the recursive calculation of the interpolating L spline.

However, processes Y which give rise to minimum variance estimators $\hat{y}$ that are L or L-g splines, as described above, are "autoregressive" in nature,

[4] Readers unfamiliar with the material in this subsection may wish to read Section 9.1 first.

that is, they are modeled by dynamical systems, driven by white noise, possessing only "denominator dynamics." More general processes Y which are both "autoregressive" and "moving average," i.e., which require both denominator and numerator dynamics in their dynamic modeling, lead to LM or LM-g splines, introduced in the present paper, as minimum variance estimators.

In the following subsections, we discuss stochastic models on the basis of which the Type I LM-g interpolating and smoothing splines are derived as least-squares estimates.

5.2. *A Stochastic Model Associated with the LM-g Interpolating Spline of Type I*

In our state-variable model (Eqs. (66a)–(66e)) for the process Y pertaining to the LM-g interpolating spline of type I, the introduction of the operator M essentially generalizes the structure of the matrix[5] c in the model derived for the L spline by Weinert and Sidhu [10].

To obtain the proposed model then, we note that, in our case, the covariance of Y is the reproducing kernel $K(t, u)$ given by (29), and proceed exactly as in [10] by partitioning $K(t, u)$ in the form

$$K(t, u) = K_0(t, u) + K_1(t, u), \tag{58}$$

where

$$K_0(t, u) = \sum_{i=1}^{n_1} \eta_i(t)\, \eta_i(u); \qquad K_1(t, u) = \int_0^T M_{(t)} M_{(u)} G_L(t, v)\, G_L(u, v)\, dv, \tag{59a,b}$$

and introducing the second-order zero-mean stochastic processes

$$Y_0 \equiv \{Y_0(t)\colon 0 \leqslant t \leqslant T\}$$

and

$$Y_1 \equiv \{Y_1(t)\colon 0 \leqslant t \leqslant T\}$$

with covariances $K_0(t, u)$ and $K_1(t, u)$, respectively. Since K_0 and K_1 are reproducing kernels for an orthogonal decomposition of H_1^{n-m}, they induce the stochastically orthogonal decomposition of Y:

$$Y(t) = Y_0(t) + Y_1(t), \qquad 0 \leqslant t \leqslant T \tag{60}$$

[5] The matrix c referred to here is the one appearing in the measurement equation such as (66b).

(where, by stochastic orthogonality, $E\{Y_0(t)\ Y_1(t)\} = 0$) with the corresponding relationship among the realizations y, y_0, and y_1 of Y, Y_0, and Y_1,

$$y(t) = y_0(t) + y_1(t), \qquad 0 \leqslant t \leqslant T. \tag{61}$$

It follows from (58), (60), and (61) that $y_0(t)$ is the least squares estimate of $y(t)$ given $\phi_i y$, $i = 1,..., n_1$ (i.e., y_0 is the projection of y on the span of the representers of ϕ_i, $i = 1,..., n_1$). Hence,

$$y_0(t) = \sum_{i=1}^{n_1} (\phi_i y)\ \phi_{i(u)} K(t, u)$$

$$= \sum_{i=1}^{n_1} (\phi_i y)\ \eta_i(t) \tag{62a}$$

$$= \sum_{i=1}^{n_1} r_i \eta_i(t). \tag{62b}$$

According to (30), (15), (22a), and (22b), (62a)-(62b) is equivalent to

$$y_0(t) = M\tilde{y}_0(t), \tag{63a}$$

$$L\tilde{y}_0(t) = 0, \tag{63b}$$

$$\phi_i M\tilde{y}_0 = r_i\ ,\ i = 1,..., n_1\ ; \qquad \psi_j \tilde{y}_0 = 0, j = 1,..., n_2\ . \tag{63c}$$

Also, from (59b) (which is the covariance of the process Y_1) it is clear that the covariance of the process $LG_M Y_1$ is $\delta(t - u)$, and hence, $y_1(t)$, $0 \leqslant t \leqslant T$, is described by

$$L_{(t)} G_M(t, \cdot) \circ y_1 = w(t), \tag{64a}$$

$$\phi_i y_1 = 0,\ i = 1,..., n_1\ ; \qquad \phi_i y_1 = r_i - \phi_i y_0 \equiv \hat{r}_i\ ;\ i = n_1 + 1,..., k, \tag{64b}$$

where $w(\cdot)$ is a sample function of white noise (formal derivative of a Wiener process) with unit impulse covariance.

The system (64a), (64b) is equivalent to

$$y_1(t) = M\tilde{y}_1(t), \tag{65a}$$

$$L\tilde{y}_1(t) = w(t), \tag{65b}$$

$$\left.\begin{array}{ll} \phi_i M\tilde{y}_1 = 0, \quad i = 1,..., n_1\ ; & \psi_j \tilde{y}_1 = 0, \quad j = 1,..., n_2\ ; \\ \phi_i M\tilde{y}_1 = r_i - \phi_i M\tilde{y}_0 \equiv \hat{r}_i\ , & i = n_1 + 1,..., k. \end{array}\right. \tag{65c}$$

Thus we conclude that the set of Eqs. (61), (63a–c), and (65a–c) provide an *input–output description* for the process Y which we are seeking. This description can be converted to any one of the standard *state variable descriptions* for systems such as the above available in the literature (see, e.g., [18]). The following particularly simple *state-variable model* results from (61), (63a–c) and (65a–c), if we assume that $m < n$:

$$\dot{\underset{\sim}{x}}(t) = A(t)\,\underset{\sim}{x}(t), \qquad \cdot = d/dt, \tag{66a}$$

$$y_0(t) = c^T(t)\,\underset{\sim}{x}(t), \tag{66b}$$

$$\dot{x}(t) = A(t)\,x(t) + bw(t), \tag{66c}$$

$$y_1(t) = c^T(t)\,x(t), \tag{66d}$$

$$y(t) = y_0(t) + y_1(t), \tag{66e}$$

where $\underset{\sim}{x}(t)$ and $x(t)$ are real n-vectors, which together express the "state" of the system at t, and $A(t)$ is an $n \times n$ matrix and b and $c(t)$ are n-column matrices described by:

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \cdot\;\cdot\;\cdot & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & -a_3(t) & \cdots & -a_{n-2}(t) & -a_{n-1}(t) \end{bmatrix}, \tag{66f}$$

$$b = \operatorname{col}(0, 0, \ldots, 0, 1), \tag{66g}$$

$$c(t) = \operatorname{col}(b_0(t), b_1(t), \ldots, b_m(t), 0, \ldots, 0). \tag{66h}$$

To complete this description, we need to specify, in terms of the data, the initial state $\underset{\sim}{x}(0)$ of (66a), the mean (which is equal to zero) and covariance of the initial state $X(0)$ of (66c) (where we denote by $X = \{X(t): 0 \leqslant t \leqslant T\}$ the stochastic process of which $\{x(t): 0 \leqslant t \leqslant T\}$ in (66c), (66d) is a realization), as well as the correlation of $X(0)$ with the input. These quantities are expressed by Eqs. (71) to (77) below, the derivations of which we omit since they constitute a straightforward application to our case of the procedure followed with the L spline in [10].

Denoting by $P(t_2, t_1)$, the principal matrix solution of (66a), and introducing the n-column matrices

$$\rho \equiv \operatorname{col}(\rho_1, \ldots, \rho_n) \equiv \operatorname{col}(\phi_1, \ldots, \phi_{n_1}, \psi_1, \ldots, \psi_{n_2}), \tag{67}$$

$$\tilde{r} \equiv \operatorname{col}(\tilde{r}_1, \ldots, \tilde{r}_n) \equiv \operatorname{col}(r_1, \ldots, r_{n_1}, 0, \ldots, 0), \tag{68}$$

the constant $n \times n$ matrix

$$F \equiv \rho_{(t)}\, c^T(t)\, P(t, 0) \tag{69}$$

(the ijth element of which consists of the action of the functional ρ_i on the function $\sum_{p=1}^{n} c_p(\cdot)\, P_{pj}(\cdot, 0)$ on $[0, T]$), and with the notation

$$\begin{aligned} P(t, v)_+ &= 0, \qquad && t < v \\ &= P(t, v), && t > v, \end{aligned} \tag{70}$$

we obtain

$$\tilde{x}(0) = F^{-1}\tilde{r} \tag{71}$$

(and hence, according to (66b))

$$y_0(t) = c^T(t)\, P(t, 0)\, F^{-1}\tilde{r}, \tag{72}$$

$$E\{X(0)\} = O, \tag{73}$$

$$E\{X(0)\, X^T(0)\} = F^{-1}C(F^T)^{-1}, \tag{74}$$

$$C = \{C_{ij}\}_{i,j=1,\dots,n}\,, \tag{75}$$

$$C_{ij} = \rho_{i(t)}\rho_{j(t')} \left[c^T(t) \left(\int_0^{\min(t,t')} P(t, s)\, bb^T P^T(t', s)\, ds\right) c(t')\right], \tag{76}$$

$$E\{X(0)\, W(v)\} = \rho_{(t)} c^T(t)\, P(t, v)_+\, b. \tag{77}$$

Now (73), (74), and (77) may be used with any standard least-squares smoothing algorithm [19], and in particular with a modified version of the algorithm in [10], for the recursive calculation of the *LM-g* spline of type I identified as the least squares estimate $\hat{y}(t)$ of $y(t)$ described by (66a–h), given the data.

5.3. *A Stochastic Model Pertaining to the LM-g Smoothing Spline of Type I*

From (44) it easily follows, by an extension of the argument presented in the preceding section, that the value $S_I(L, M, \Psi, \Phi, r; Q; t)$ of the *LM-g* smoothing spline of type I is the least-squares estimate of $y(t)$ described by (61), (63), and (65), with the first equation in (63c) and (65c) replaced respectively by

$$\phi_i M\tilde{y}_0 + z_i = r_i\,, \quad i = 1,\dots, n_1\,;$$

$$\phi_i M\tilde{y}_1 + z_i = r_i - \phi_i M\tilde{y}_0 \equiv \hat{r}_i\,, \quad i = n_1 + 1,\dots, k,$$

where $z = \operatorname{col}(z_1\,,\dots, z_k)$ is a measurement noise vector with covariance Q and independent of y.

The corresponding *state variable model* is again described by (66a)–(66h) with appropriate accounting of the initial conditions. However, in the ensuing least-squares smoothing algorithm for the recursive calculation of

$\hat{y}(t)$ (which is equal to $S_I(L, M, \Psi, \Phi, r; Q; t)$), the covariance Q of the additive measurement noise z (indicated above) has to be included in a standard way [19].

6. SUMMARY OF RESULTS FOR INTERPOLATING AND SMOOTHING LM-g SPLINES OF TYPE II

Let L and M be the differential operators described in Section 2 except that now[6] the coefficients a_j and b_i satisfy, respectively, $a_j \in C^{j+m}[0, T]$, $j = 0,\ldots, n-1$, and $b_i \in C^i[0, T]$, $i = 0,\ldots, m$. Thus L and M are continuous linear operators, respectively from H^{n+m} to H^m and H^m to $L^2(0, T)$.

Suppose that we are given a set ψ of linear functionals ψ_i, $i = 1,\ldots, m$, on H^m which are linearly independent on $N(M)$ and a set Φ of linearly independent functionals ϕ_j, $j = 1,\ldots, k$, $k \geqslant n$, on H^n (and hence also on H^{n+m}) such that $\phi_1,\ldots, \phi_n$, constituting a subset $\tilde{\Phi}$ of Φ, are linearly independent on $N(L)$. Here, $\Psi \in \mathscr{F}^m$ and $\Phi \in \mathscr{F}^n$. It is then possible to define

$$H_\psi{}^m = \{g \in H^m: \psi_i g = 0, i = 1,\ldots, m\}, \tag{78a}$$

$$\tilde{H}_\Phi^{n+m} = \{g \in H^{n+m}: \phi_i g = 0, i = 1,\ldots, n\}, \tag{78b}$$

and

$$\tilde{H}_\Phi{}^n = \{g \in H^n: \phi_i g = 0, i = 1,\ldots, n\}. \tag{78c}$$

In a way analogous as before we may introduce the Green's functions $G_M : MH_\psi{}^m \to H_\psi{}^m$ and $G_L : L\tilde{H}_\Phi{}^n \to \tilde{H}_\Phi{}^n$. Note that G_L is well defined on $L\tilde{H}_\Phi^{n+m}$.

Let

$$H_1^n \equiv G_L(ML\tilde{H}_\Phi^{n+m} \cap L\tilde{H}_\Phi{}^n) \oplus N(L), \tag{78d}$$

and, given a data k vector r (as in (39)), let

$$U_2(r) = \{f \in H_1{}^n: \phi_i f = r_i, i = 1,\ldots, k\}. \tag{79}$$

We then have:

DEFINITION 5. With L, M, Ψ, Φ, $\tilde{\Phi}$, and r as above, an LM-g spline $S_{II}(L, M, \Psi, \Phi, r; \cdot)$ of type II interpolating r with respect to Φ is defined by

$$\min_{f \in U_2(r)} \int_0^T (G_M(t, \cdot) \circ Lf)^2 \, dt = \int_0^T (G_M(t, \cdot) \circ LS_{II}(L, M, \Psi, \Phi, r; \cdot))^2 \, dt. \quad \| \tag{80}$$

The following results can be proved by the methods of Section 3.2.

[6] In this section as well as in the following one, some of the previously used symbols are redefined to fit the definition and developments related respectively to LM-g splines of types II and III.

THEOREM 8. *The LM-g interpolating spline of Definition 5 exists, and, in view of the way we defined Φ, it is unique.* ||

Let η_i, $i = 1,..., n$, constitute the basis for $N(L)$ dual to ϕ_i, $i = 1,..., n$, and use the notation $\hat{H}^{n+m} \equiv G_L G_M \, L H_1^n$.

THEOREM 9. *$S_{II}(L, M, \Psi, \Phi, r; \cdot)$ satisfies*

$$S_{II}(L, M, \Psi, \Phi, r; t) = G_L(t, \cdot) \circ ML\tilde{f}^*, \tag{81}$$

where $\tilde{f}^*$ is the L-g spline defined by

$$\min_{\substack{\tilde{f} \in \hat{H}^{n+m} \\ \tilde{\Phi}_i \tilde{f} = \phi_{i(t)} G_L(t,.) ML\tilde{f} = r_i, i=1,...,k}} \left\{ \int_0^T (L\tilde{f}(t))^2 \, dt \right\} = \int_0^T (L\tilde{f}^*(t))^2 \, dt. \quad \| \tag{82}$$

THEOREM 10. *H_1^n, equipped with the inner product*

$$\langle f, g \rangle = \sum_{i=1}^{n} (\phi_i f)(\phi_i g) + \int_0^T [G_M(t, \cdot) \circ Lf][G_M(t, \cdot) \circ Lg] \, dt \qquad \forall f, g \in H_1^n, \tag{83}$$

constitutes a reproducing kernel Hilbert space with the reproducing kernel

$$K(t, u) = \sum_{i=1}^{n} \eta_i(t) \, \eta_i(u) + \int_0^T [M_{(v)}^{\dagger} G_L(t, v)][M_{(v)}^{\dagger} G_L(u, v)] \, dv, \tag{84}$$

where the superscript † denotes the adjoint. ||

THEOREM 11. *$S_{II}(L, M, \Psi, \Phi, r; \cdot)$ is the solution of the minimum norm problem in H_1^n (with the norm $\| \cdot \|$ induced by the inner product* (83))

$$\min_{f \in U_2(r)} \|f\|^2. \quad \| \tag{85}$$

It is clear that, provided ϕ_j and K are interpreted as in the present section, formula (41) also permits an explicit representation for the *LM-g* spline in the present case.

Finally, as in Definition 4, we have:

DEFINITION 6. If Q and r are as in Definition 4 and L, M, Ψ, and Φ as above, *an LM-g smoothing spline $S_{II}(L, M, \Psi, \Phi, r; Q; \cdot)$ of type II* is the solution of the problem

$$\min_{f \in H_1^n} \left\{ \int_0^T (G_M(t, \cdot) \circ Lf)^2 \, dt + (r - \Phi f)^T Q^{-1} (r - \Phi f) \right\}. \quad \| \tag{86}$$

Again, in the reproducing kernel Hilbert space of Theorem 10 a representation for $S_{II}(L, M, \Psi, \Phi, r; Q; \cdot)$ can be obtained exactly as for $S_I(L, M, \Psi, \Phi, r; Q; \cdot)$ in Definition 4.

Analogous to the system (61), (63), and (65), we have the following *input-output* stochastic model (consisting of (87), (88), and (89)) for which the LM-g spline of type II is the least-squares estimate:

$$y(t) = y_0(t) + y_1(t), \tag{87}$$

$$Ly_0(t) = 0, \tag{88a}$$

$$\phi_i y_0 = r_i, \qquad i = 1,..., n, \tag{88b}$$

$$Ly_1(t) = Mw(t), \tag{89a}$$

$$\phi_i y_1 = 0,\ i = 1,..., n; \qquad \phi_j y_1 = r_j - \phi_j y_0 \equiv \hat{r}_j,\ j = n+1,..., k. \tag{89b}$$

Remark 3. Equation (87) is a realization of an orthogonal decomposition of the process Y as in (60), with the covariances of Y_0 and Y_1 equal, respectively, to the summation and integral terms in (84). We will simply indicate how (89a) is obtained. From the definition of the process Y_1, it follows that the covariance of the process $G_M LY_1$ is

$$\begin{aligned} K_{1G_ML}(t, u) &= \int_0^T \left[\int_0^T G_M(t, t')\, L_{(t')}M^{\dagger}_{(t')}G_L(t', v)\, dt'\right] \\ &\quad \times \left[\int_0^T G_M(u, u')\, L_{(u')}M^{\dagger}_{(u')}G_L(u', v)\, du'\right] dv \\ &= \int_0^T \left[\int_0^T G_M(t, t') \int_0^T L_{(t')}G_L(t', s)\, M_{(s)}\delta(s - v)\, ds\, dt'\right] \\ &\quad \times \left[\int_0^T G_M(u, u') \int_0^T L_{(u')}G_L(u', z)\, M_{(z)}\delta(z - v)\, dz\, du'\right] dv \\ &= \int_0^T \left[\int_0^T G_M(t, t')\, M_{(t')}\delta(t' - v)\, dt'\right] \\ &\quad \times \left[\int_0^T G_M(u, u')\, M_{(u')}\delta(u' - v)\, du'\right] dv \\ &= \int_0^T (M^{\dagger}_{(v)}G_M(t, v))(M^{\dagger}_{(v)}G_M(u, v))\, dv \\ &= \int_0^T \delta(t - v)\, \delta(u - v)\, dv = \delta(t - u), \end{aligned} \tag{90}$$

which is the covariance of white noise $w(t)$. This leads to the model:

$$G_M(t, \cdot) \circ Ly_1 = w(t), \tag{91}$$

which, when formally operated on both sides by M, leads to (89a). ‖

From elementary considerations [18] it follows that Eqs. (66a–e) constitute a valid *state-variable model* for the system (87)–(89), with the matrix $A(t)$ defined by (66f), and Eqs. (66g) and (66h) replaced by (66g′) and (66h′) below:

$$b(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_{n-1}(t) & 1 & 0 & 0 & \cdots & 0 \\ a_{n-2}(t) & a_{n-1}(t) & 1 & 0 & \cdots & 0 \\ & & \cdots & & & \\ a_0(t) & a_1(t) & a_2(t) & a_3(t) \cdot a_{n-1}(t) & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m(t) \\ b_{m-1}(t) \\ \cdot \\ b_0(t) \end{bmatrix} \tag{66g′}$$

(where the rightmost member is an n-vector consisting of the coefficients $b_i(t)$, $i = 0,\ldots, m$, of M and $(n - m - 1)$ zeros, as indicated),

$$c(t) \equiv c = \mathrm{col}(1, 0,\ldots, 0). \tag{66h′}$$

Conditions (73), (74), and (77) also apply to the present case, provided b and c are defined as in (66g′) and (66h′), and (67) and (68), are:

$$\rho = \mathrm{col}(\phi_1 ,\ldots, \phi_n), \tag{67′}$$

$$\tilde{r} = \mathrm{col}(r_1 ,\ldots, r_n). \tag{68′}$$

Finally, it is clear that the above model also leads to the LM-g smoothing spline of type II as a least-squares estimate provided (89b) is replaced by

$$\phi_i y_0 + z_i = r_i , \; i = 1,\ldots, n; \qquad \phi_j y_1 + z_j = r_j - \phi_j y_0 \equiv \hat{r}_j , \; j = n + 1,\ldots, k \tag{89b′}$$

7. LM-g Splines of Type III

A third type of LM-g spline may be introduced if we assume L to be the same as for S_I, and the coefficients of M to be such that M is an operator from H^{n+m} to H^n. Let n_4 where $n \leqslant n_4 \leqslant n + m$, denote the dimension of $N(LM)$. Then given a set $\Phi = \{\phi_1 ,\ldots, \phi_k\}$, where $k \geqslant n_4$, of linear functionals in $\mathscr{F}^{n+m}$, (with $\phi_1 ,\ldots, \phi_m$ linearly independent on $N(M)$ and $\phi_1 ,\ldots, \phi_{n_4}$ linearly independent on $N(LM)$) and a real k-vector r, we have:

Definition 7. An *LM-g spline $S_{III}(L, M, \Phi, r; \cdot)$ of type III interpolating r with respect to Φ* is described by

$$\min_{f \in U_3(r)} \int_0^T (LMf(t))^2 \, dt = \int_0^T (LMS_{III}(L, M, \Phi, r; t))^2 \, dt \tag{92}$$

where

$$U_3(r) = \{f \in H^{n+m}: \phi_i f = r_i, i = 1,..., k\}. \quad \| \tag{93}$$

It follows that an *LM-g* spline of type III is an *L-g* spline with the differential operator consisting of the composition *LM*. However, its structure does complete the picture of classes of generalized splines generated by two differential operators, and for this reason we bring it up here. In fact, if we define the Green's function G_M pertaining to M and to the set of functionals $\phi_1,..., \phi_m$, and introduce the set $\Gamma \equiv \{\gamma_1,..., \gamma_k\} = \{\tilde{\phi}_1,..., \tilde{\phi}_k\}$ of linear functionals on H^n defined by

$$\tilde{\phi}_i = \phi_i G_M, \qquad i = 1,..., k, \tag{94}$$

then, analogous to the result (20) for the *LM-g* spline of type I, we have in the present case

$$S_{III}(L, M, \Phi, r; t) = G_M(t, \cdot) \circ \tilde{S}(L, \Gamma, q; \cdot) \tag{95}$$

where $q = r$, and $\tilde{S}(L, \Gamma, q; \cdot)$ is the *L-g* spline in a subspace of MH^{n+m} interpolating q with respect to Γ. It is clear (and hence we do not further elaborate it here) that developments analogous to those in sections 2 to 5 hold in the present case with the role of M in those sections replaced by G_M.

8. The Case in Which L and M are Constant Coefficient Operators

It is of interest to note that if L and M are constant coefficient differential operators, then L and M commute in the integrals appearing in (10) and (44) provided the conditions of the following theorem are satisfied. It follows that in such a case the *LM-g* splines of types I and II are the same.

THEOREM 12. *Suppose L and M are constant coefficient differential operators, and either (a) $U_1(r)$ is restricted to the subspace of $MH_\psi{}^n$ consisting of functions f which vanish at end-points (this is equivalent to requiring that two of the constraining functionals ϕ_i, say ϕ_1 and ϕ_2, satisfy $\phi_1 f \equiv f(0)$, $\phi_2 f \equiv f(T)$, and $r_1 = r_2 = 0$); or (b) $\phi_1 f \equiv f(0) = r_1 \equiv 0$ and $\psi_1 \tilde{f} \equiv \tilde{f}(0),...,$ $\psi_m \tilde{f} \equiv \tilde{f}^{(m-1)}(0)$. (This requirement on ψ_i, $i = 1,..., m$, is equivalent to the condition that G_M be causal.) Then L and G_M commute in* (10) *and* (44).

Proof. In (10) and (44) we have

$$\begin{aligned} L_{(t)}G_M(t, \cdot) \circ f &= \int_0^T L_{(t)}G_M(t-s) f(s)\, ds \\ &= \int_0^T L^\dagger_{(s)}G_M(t-s) f(s)\, ds \\ &= \int_0^T G_M(t-s) L_{(s)} f(s)\, ds - G_M(t-T) f(T) + G_M(t) f(0), \end{aligned} \tag{96}$$

the last equality following from Green's formula [20, p. 86].

From (96) and either of the sets of conditions in the theorem, the result of the theorem follows. ||

9. Applications to Signal and System Theory Problems

9.1. *Estimation of Mixed Stochastic Processes*

As is wellknown, in the solution of the optimal linear filtering and prediction problems, the structure of the filter or predictor is based on the models assumed for the signal and noise stochastic processes. For any such stochastic process[7] $Y \equiv \{Y(t): 0 \leqslant t \leqslant T\}$, the model most commonly used assumes Y to be the output of a linear dynamical system $\mathscr{G}$ driven by white noise $W = \{W(t): 0 \leqslant t \leqslant T\}$.

If the process Y is stationary, $\mathscr{G}$ is time-invariant and can therefore be described by a "transfer function" $G(i\omega)$, where $i = (-1)^{1/2}$ and ω is the frequency variable. A reasonable assumption to make in most cases is that $\mathscr{G}$ is lumped. This is equivalent to assuming that G is a rational function, that is,

$$G(i\omega) = M(i\omega)/L(i\omega), \tag{97a}$$

where L and M are polynomials in $i\omega$ of the form

$$L(i\omega) = (i\omega)^n + \sum_{j=0}^{n-1} a_j(i\omega)^j, \tag{97b}$$

$$M(i\omega) = \sum_{p=0}^{m} b_p(i\omega)^p, \tag{97c}$$

a_j , $j = 0,\ldots, n-1$, and b_p , $p = 0,\ldots, m$, being constants.

A process Y modeled as above is depicted in Fig. 1. For white noise with covariance $K_W(t, s) = \delta(t - s)$ (and hence with unit spectral density), the spectral density $S_Y(\omega)$ of Y is

$$S_Y(\omega) = | G(i\omega)|^2 = \frac{| M(i\omega)|^2}{| L(i\omega)|^2}. \tag{98}$$

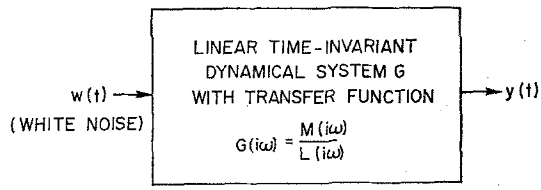

FIG. 1. Model of a stationary stochastic process Y.

[7] As done earlier in the text, random processes will be denoted by capital letters and their sample functions by the corresponding small letters.

In the above formulation, the polynomials L and M are said to express respectively the "denominator" and "numerator" dynamics of the system $\mathscr{G}$. Processes Y for which the degree m of M is zero (and hence for which G consists only of denominator dynamics) are called *autoregressive processes.* On the other hand, processes for which G consists only of numerator dynamics ($n = 0$) (with differentiation of white noise appropriately interpreted) are called *moving average processes*. Processes for which both numerator and denominator dynamics are present are called *mixed processes.*

It is clear that the above classification of stochastic processes Y holds if we generalize the structure of $\mathscr{G}$ to that of a linear time-varying system described by a differential equation of the form

$$Ly(t) = Mw(t), \tag{99}$$

where L and M are the differential operators defined by (1) and (2). According to the Fourier transform theory, in the special case in which L and M in (99) have constant coefficients, the differential equation description (99) of $\mathscr{G}$ corresponds to the description by means of the transfer function $G(i\omega)$ given by (97).

Let us now return to the spline approximation problem. Because it is easier to visualize the underlying ideas in terms of transfer functions, we shall assume in the following that the differential operators L and M have constant coefficients. Our remarks will clearly extend to the case in which they vary with t.

The *interpolation* by one of the three types of *LM-g splines* proposed by us corresponds to the least squares *noiseless prediction* of a mixed process modeled as in Fig. 1, while the *smoothing by such an LM-g spline* corresponds to the least squares *filtering or prediction* of such a process in the presence of observation (measurement) noise.

Specifically, in the *interpolation problem*, the function $y(\cdot)$ to be interpolated is given by (61), with $y_0(\cdot)$ a suitably defined function dependent on some of the data, as indicated before, and $y_1(\cdot)$ a sample function of a random process Y_1 described by one of the models represented in Figs. 2, 3, or 4, depending on whether the interpolation is by an *LM-g* spline of type I, II, or III. Note that the difference between the models pertaining respectively to *LM-g* splines to types I and II lies in whether the numerator dynamics follows or precedes the denominator dynamics.

For the cases in Figs. 2, 3, and 4, the least-squares predictor of $y(t)$ based on the discrete noiseless measurements $\phi_i y = r_i$, $i = 1,..., k$ are, respectively $\hat{y}(t) = S_j(L, M, \Psi, \Phi, r; t), j = I, II$, and $S_{III}(L, M, \Phi, r; t)$.

In the smoothing problem, the signal process Y is still described as in Figs. 2, 3, and 4, but the discrete measurements are corrupted by additive noise z_i, $i = 1,..., k$. The least-squares estimate of $y(t)$ is then $\hat{y}(t) = S_j(L, M, \Psi, \Phi, r; Q; t), j = I$ or II, or $S_{III}(L, M, \Phi, r; Q; t)$.

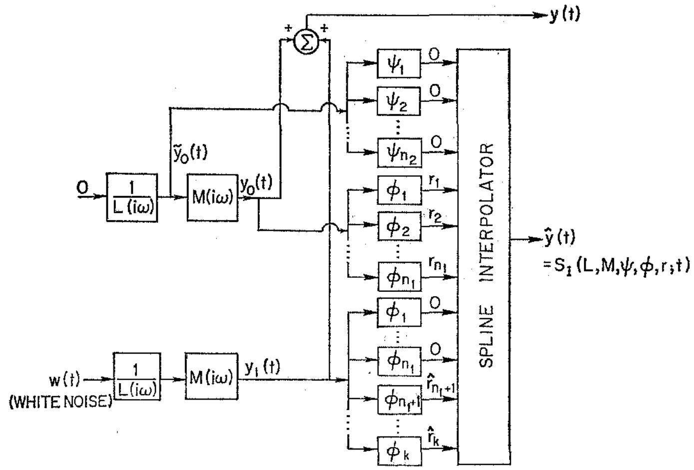

FIG. 2. Stochastic model pertaining to an *LM-g* interpolating spline of type I.

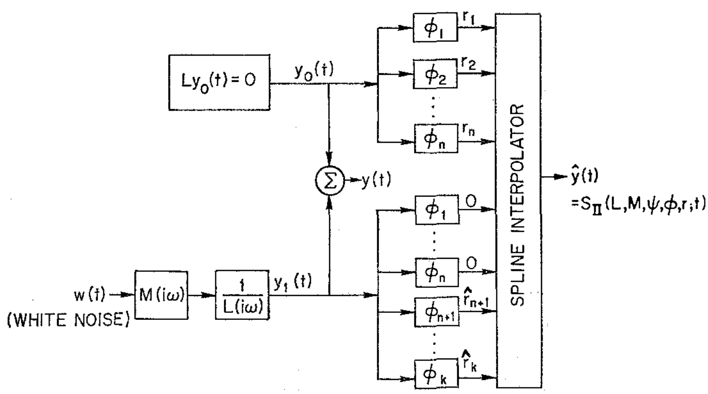

FIG. 3. Stochastic model pertaining to an *LM-g* interpolating spline of type II.

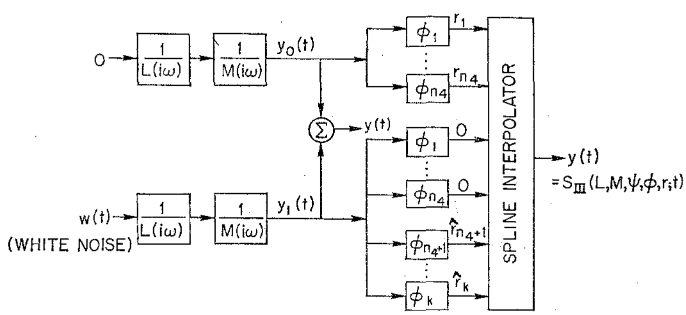

FIG. 4. Stochastic model pertaining to an *LM-g* interpolating spline of type III.

It is clear then that on both the interpolation and smoothing problems, the least-squares estimate $\hat{y}(t)$ may be obtained by means of a recursive least-squares algorithm [18].

9.2. *Minimum Energy Control Problems*

The way in which generalized splines appear in the formulation of the solution to a class of minimum energy control problems has been discussed in [21, 8, 11, 22]. However, the systems considered in those references possess no numerator dynamics. The *LM-g* splines introduced here not only permit the generalization of these results to systems with both numerator and denominator dynamics but also bring further insights into the existing results. We shall first explain the second part of the last statement, assuming, for simplicity in presentation, that the constraining functionals are interpolating.

The question under consideration may be posed as follows:

Problem 1. Given a dynamical system whose input u and output y are related by

$$Ly(t) = u(t), \qquad 0 \leqslant t \leqslant T, \tag{100}$$

where L is the differential operator in (1), and subject to the output constraints

$$y(t_i) = r_i, \qquad i = 1,..., k,\ t_i \in \Delta, \tag{101}$$

where Δ: $0 < t_1 < \cdots < t_k < T$, and r_i, $i = 1,..., k$, are real numbers; find the control u^* in the range of L of minimum energy, that is, which minimizes the $L^2(0, T)$ norm of u under (100) and (101). ||

According to the minimum norm property of L splines [13], the y^* which when replaced in (100) gives u^*, is the L-spline $y^*(t) = \tilde{S}(L, \Delta, r; t)$, i.e., a y^* satisfying:

$$\text{(a)}\quad L^\dagger Ly^*(t) = 0, \qquad t_i \leqslant t < t_{i+1},\ i = 1,..., N-1; \tag{102a}$$

$$\text{(b)}\quad y^*(t_i) = r_i, \qquad t_i \in \Delta; \tag{102b}$$

$$\text{(c)}\quad y^* \in C^{2n-2}; \tag{102c}$$

$$\text{(d)}\quad Ly^*(t) = 0, \qquad 0 \leqslant t < t_1, \text{ and } t_N \leqslant t \leqslant T. \tag{102d}$$

Applying (100) to (102) we conclude that the u^* that we are seeking is defined by

$$u^*(t) = L\tilde{S}(L, \Delta, r; t), \tag{103}$$

which according to Theorem 1 νhows that u^* is an LM spline of type I with $M = L$.

As a generalization of Problem 1 we have:

Problem 2. Same as Problem 1 with Eq. (100) replaced by

$$Ly(t) = Mu(t), \qquad 0 \leqslant t \leqslant T, \tag{100'}$$

where M is as in (2). ||

It is clear that the function y^* satisfying (100′) and the remaining conditions of Problem 2 (instead of being defined by (102a–d)) is an appropriate LM spline of type II, $S_{II}(L, M, \Delta, r; t)$. Also, the application of the operator $G_M L$ to (81) shows (in light of (100′)) that u^* in this case is again an LM spline of type I.

As we have said before, the remarks in this subsection extend immediately to the case in which the evaluation functionals are replaced by arbitrary continuous linear functionals, this bringing into the discussion L-g and LM-g splines rather than L and LM splines.

9.3. *Digital Signal Processing and System Modeling*

An approach to the optimal design of digital filters and digital simulators, based on an appropriate modeling of the signal source and a subsequent use of the theorem on best approximation of linear functionals [16], was presented in [23]. This approach was further extended in [24] to the modeling of systems on the digital computer. In these works, generalized splines provide a natural setting for the formulation and solution of the problems posed.

The LM-g splines introduced in the present paper permit us to generalize the source models previously used. Specifically, the model used in [23] for the signal source is of the form

$$S = \{f \in H^n : \| Lf \|^2 \leqslant \gamma^2, \ \phi_i f = r_i, \ i = 1,..., k.\} \tag{104}$$

where the meaning of the symbols is clear. The generalization of (104) referred to above is obtained by replacing Lf in (104) by $LG_M f$, $G_M Lf$, or LMf.

10. Conclusion

Three types of genralized splines, called LM-g splines of types I, II, and III, have been introduced as a generalization of the L-g spline [2]. Their properties have been investigated and their role in problems of estimation of stochastic processes, minimum energy controls, and digital signal processing and system modeling indicated. Further discussion on the algorithms that result from the present formulation is contained in [17, 25].

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